J. Fluid Mech. (1995), vol. 290, pp. 299–317

On vortex air motions above an axisymmetric source of mass, momentum and heat

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(Received 22 April 1994 and in revised form 10 December 1994)

We study axisymmetric plume dispersion from a steady source of mass, momentum and/or heat that is subjected to either a time-dependent large-scale external vortex or small-scale turbulent axisymmetric helicity. On the basis of the turbulent boundary layer and Boussinesq assumptions and by assuming similarity profiles with Gaussian distribution in the radial direction the balance equations of mass, momentum, and energy reduce to a system of nonlinear differential equations for amplitude functions of axial velocity, pressure and density differences as well as azimuthal velocity. The system of equations is closed with Taylor's entrainment assumption.

The plume radius and the typical radius of the large-scale external vortex are also determined. For a simple density structure of the ambient atmosphere (i.e. adiabatic conditions) analytical results can be obtained, but for more complicated cases, i.e. a layered polytropic atmosphere, the governing equations are examined numerically; computations are reasonably simple and efficient.

1. Introduction

The study of air flows induced by mass and heat sources of different types, such as forest and urban fires, industrial and volcanic eruptions, explosions etc., is of great interest for the evaluation of their environmental impact. If one knows the characteristics of the propagation of heat and particle concentration in the atmosphere due to such sources, it appears likely that the consequences of dramatic events can be predicted in the immediate vicinity above the source and some protective measures can be taken. Besides their practical importance, such investigations also provide fundamental information because of the complexity of the physical and chemical processes that are involved. Axisymmetric air flows produced by fire and known as convective columns or 'fire plumes', in which hot or warm air rises vertically above the combustion zone, are accompanied by radial inflows from the ambient atmosphere. Furthermore, highly complicated interactions of buoyancy and dynamic pressure changes take place. Such phenomena have been extensively studied. An early general study is by Long (1967), a more recent one by Carrier, Fendall & Feldman (1985); these may be complemented by works of Smith, Morton & Leslie (1978), Grishin, Gruzin & Zverev (1983), Small & Larson (1984/85), Markatos & Pericleous (1984), Turner (1986), Small & Heikes (1988) and Gostintsev *et al.* (1991). One approach is Taylor's entrainment theory (see Turner 1986) which permits the determination of the integral characteristics of the plumes. The theory is based on the following entrainment assumption: there is an ambient air inflow with a mean velocity that is proportional to the axial plume velocity averaged over the cross-section at the inflow level. Such a model cannot accurately describe the flow field near the heat and/or momentum source at the ground level, nor the flow in the immediate vicinity of the combustion zone. However, it can be considered to be a valid approximation at appreciable height above the combustion zone. There are many geophysical examples demonstrating the practical success of the entrainment assumption (Turner 1986).

Another approach is based on the numerical modelling of the full three-dimensional problem; it permits calculation of the hydrothermodynamic processes in and around the combustion zone, see e.g. Smith *et al.* (1975), Small & Larson (1984/85), Small & Heikes (1988), Markatos & Pericleous (1983), Grishin *et al.* (1983) and Gosthintsev *et al.* (1991). In this approach it is equally necessary to introduce simplifying assumptions because the full dynamical problem must be described by coupled equations of hydro- and combustion dynamics. In the case that the accompanying density and temperature changes can be considered small, the Boussinesq approximation can be employed, (Smith *et al.* 1975). If they are large, the Boussinesq approximation is hardly valid; in this case some authors consider the flows above a source as those of an ideal compressible gas, the density of which does not depend on pressure. Combustion kinetics is replaced by some volume heat addition (Small & Larson 1984/85; Small & Heikes 1988). There is only a small number of articles that take into account (i) the chemical kinetics of combustion products, (ii) higher-order complex turbulent closure hypotheses for the viscosity and heat conductivity and (iii) phase transitions (Grishin *et al.* 1983; Markatos & Pericleous 1984; Gostintsev *et al.* 1991). All these investigations make use of very large amounts of numerical calculations.

Most articles dealing with fire plumes are devoted to non-rotating structures. Some observations, however, show that rotating convective columns do sometimes exist (Carrier *et al.* 1985; Long 1967). There are at least two possibilities for why the plumes rotate. The first is considered by Carrier *et al.* (1985) and is related to external large-scale vortices which initiate and draw convective columns into rotation. In this article we study such a model at greater length. The second one, which we propose here and consider in detail, is associated with the formation and existence of large-scale vortical structures in the presence of small-scale helical turbulence (see Berezin, Hutter & Zhukov 1991) and the references quoted therein. Hence, in this work we analyse rotating fire plumes on the basis of continuum mechanics.

2. Rotating plumes associated with external large-scale vortices

In this section we derive more precisely and amend in certain respects the equations given in Carrier *et al.* (1985) and represent the results and analysis of their solutions.

2.1. *Assumptions and equations*

The governing equations are the balance laws of mass, radial momentum, axial momentum and energy in steady form; azimuthal angular momentum in unsteady form; and the equation of state. Axisymmetric conditions are implied and no dissipative processes are assumed:†

$$(r\rho u)_r + (r\rho w)_z = 0, \tag{2.1a}$$

$$(r\rho u^2)_r + (r\rho uw)_z + rp_r - \rho v^2 - 2r\Omega\rho v = 0, \tag{2.1b}$$

$$r^2(\rho v)_t + (r^2\rho uv)_r + (r^2\rho wv)_z + 2r^2\Omega\rho u = 0, \tag{2.1c}$$

† In equations (2.1) the radial stresses τ_{rz} and heat flux q_r could be incorporated as those flux components that survive when the classical boundary layer assumption is imposed. We omit them because equations will be integrated in the radial direction later on, whereupon contributions from them vanish at $r = 0$ and $r = \infty$.

$$(r\rho uw)_r + (r\rho w^2)_z + rp_z + r\rho g = 0, \tag{2.1d}$$

$$(r\rho uT)_r + (r\rho wT)_z + r\rho gw/c_p = 0, \tag{2.1e}$$

$$p = \bar{R}\rho T. \tag{2.1f}$$

The set (2.1) is complemented by the hydrostatic equation for the still, ambient air, i.e.

$$(p_a)_z = -g\rho_a, \tag{2.2}$$

relating the vertical pressure gradient and mass density of the ambient atmosphere. In the above u, v, w are the physical velocity components in the radial, azimuthal and axial directions, respectively; $c_p = \text{const}$ is the specific heat of air at constant pressure; $\bar{R} = R/M$, where R is the universal gas constant and M the mol mass; and Ω the normal component of the Earth's angular velocity; a subscript a denotes physical characteristics of the ambient atmosphere. Equation (2.1*a*) is the steady equation of balance of mass written in axisymmetric cylindrical coordinates, (2.1*b–d*) are the radial, azimuthal and axial components of the momentum balance in which the Coriolis terms are incorporated, and (2.1*e*) is the energy equation. Indeed, if we combine the (dissipation free) enthalpy balance $\rho r\dot{h} = \dot{r}\dot{p}$, where $\dot{h} = c_p\dot{T}$ and $\dot{p} = (\mathrm{d}p/\mathrm{d}z)\dot{z} = -\rho gw$, via the hydrostatic pressure assumption, with the mass balance $\dot{\rho}rh = 0$, i.e. add the two, we obtain $(\rho rh)^{\cdot} = -r\rho gw$, whose steady axisymmetric form is given as (2.1*e*).

In the context of (2.1), (2.2) the only reason for convective column rotation is an external, strong enough, large-scale vortex. Hence, as done by Carrier *et al.* (1985) we only retain the time derivative in the azimuthal angular momentum equation. We assume also that the perfect gas law holds, and this assumption implies the equation of state, (2.1*f*).

The processes we want to study are expected to be essentially subsonic flows for which velocities are very much smaller than the velocity of sound ($|\boldsymbol{u}| \ll c_s$); it therefore is natural to assume that the mass density does not depend on pressure, e.g. density changes are associated (primarily) with temperature changes. Thus, deviations of the density and temperature fields from values corresponding to those in the ambient atmosphere, as well as velocities, are of first order, while pressure deviations from the hydrostatic value p_a are of second order in the small parameter u/c_s, $c_s = (p_0/\rho_0)^{1/2}$ where a subscript zero denotes a constant reference value (for example, the ground level value).

Thus, to leading order the equation of state takes the form $\bar{R}\rho T = p_a(z)$, and the mass density ρ may be approximated by its ambient value $\rho_a(z)$ everywhere except in the gravity term in (2.1*d*). If the spatial scales of the outside vortex are not larger than 100 km, the effect of the Earth's rotation on the vortex development can be ignored, whence the Coriolis term can be omitted, and thence we have the following equations:

$$(r\rho_a u)_r + (r\rho_a w)_z = 0, \tag{2.3a}$$

$$(r\rho_a u^2)_r + (r\rho_a uw)_z + rp_r - \rho_a v^2 = 0, \tag{2.3b}$$

$$r^2\rho_a v_t + (r^2\rho_a uv)_r + (r^2\rho_a wv)_z = 0, \tag{2.3c}$$

$$(r\rho_a uw)_r + (r\rho_a w^2)_z + rp_z + r\rho g = 0, \tag{2.3d}$$

$$(r\rho_a uT)_r + (r\rho_a wT)_z + r\rho_a gw/c_p = 0. \tag{2.3e}$$

The temperature may be expressed in terms of the density via the equation of state

$$T = T_a(1 + (\rho_a - \rho)/\rho_a).$$

It will be assumed that the ambient values of the pressure p_a, temperature T_a and mass density ρ_a are functions of the height z alone. Solutions of (2.3) are sought in similarity form using the following Gaussian profiles:

$$w(r,z,t) = W(z,t)\,e^{\psi}, \tag{2.4a}$$

$$p_a(z)-p(r,z,t) = \sigma(z,t)\,e^{\psi}, \quad \rho_a(z)-\rho(r,z,t) = f(z,t)\,e^{\psi}, \tag{2.4b, c}$$

$$v(r,z,t) = v_1(r,z,t)+v_2(r,z,t), \tag{2.4d}$$

$$v_1(r,z,t) = \frac{1}{r}\Gamma(z)(1-e^{\xi}), \quad v_2(r,z,t) = \frac{rV(z,t)}{b(z,t)}e^{\psi}, \tag{2.4e,f}$$

$$\psi(r,z,t) = -\frac{r^2}{b^2(z,t)}, \quad \xi(r,z,t) = -\frac{r^2}{B^2(z,t)}. \tag{2.4g, h}$$

Here, $W(z,t)$, $\sigma(z,t)$, $f(z,t)$, $V(z,t)$, $b(z,t)$ and $B(z,t)$ are regarded as unknown, but (2.3) form only five equations from which these can be determined. Orders of magnitude arguments are used by which the azimuthal component of the momentum equation can be split into two relations.

In (2.4), $b(z,t)$ is a radial scale for the plume characteristics, $B(z,t)$ is a typical radius of the external large-scale vortex with z-dependent circulation whose value at $r = \infty$ equals $\Gamma(z)$ (we absorb the factor $(\frac{1}{2}\pi)$ in $\Gamma(z)$); v_1 is the response to the circulation $\Gamma(z)$, whereas v_2 is the redistribution of the azimuthal component of velocity along the plume axis.

This decomposition has already been suggested by Carrier *et al.* (1985) and is implemented here within the framework of Gaussian profiles: v_1 is the simplest smooth interpolation between $v_1 = 0$ on the axis of the cylindrical frame of reference (at $r = 0$) and $v_1 \rightarrow \Gamma/r$ as $r \rightarrow \infty$; v_2 is a smooth interpolation between the zero values at $r = 0$ and $r \rightarrow \infty$ whereby the decay rate is exponential as r becomes infinitely large. The choice supposes that the only reason for a convective column rotation is an external large-scale vortex, the radius of which possesses a typical decay rate time that is substantially larger than a typical time of the unsteady processes in the plume. Such an assumption dictates the functions v_{1t} and v_{2r} to be leading-order terms in the azimuthal momentum equation. The terms of second order would yield the equation containing the z-derivatives (note that these are weaker than the r-derivatives). Incidentally, this scaling is also justified because with its typical values of the rotation times and azimuthal velocities of plumes are obtained as substantiated by measurements during the first storm in Hamburg, see Carrier *et al.* (1985).

Now, let us substitute the functions (2.4) into (2.3) and integrate them over the radial direction from $r = 0$ to ∞. At the plume axis $r = 0$ symmetry conditions must be imposed. At infinity ($r = \infty$) Taylor's entrainment assumption is invoked: the specific inflow volume flux at the 'edge' of the plume equals a fraction α of the upward axial velocity maximum, e.g. $\lim_{r\rightarrow\infty}(ru) = -\alpha bW$, where $\alpha = 0.093$ for non-rotating plumes and can be less for rotating plumes, see Carrier *et al.* (1985). Such a procedure leads to the following equations (see the Appendix for details):

$$(\rho_a b^2 W)' = 2\alpha\rho_a bW, \tag{2.5a}$$

$$(\rho_a b^2 W^2)' = 2(gfb^2+(\sigma b^2)'), \tag{2.5b}$$

$$(c_p T_a b^2 fW)' = -\frac{2g(1-k)}{k(\gamma-1)}\rho_a b^2 W, \tag{2.5c}$$

$$\sigma b^2 = 2\rho_a\left\{(2-\surd 2)\frac{b}{B}\Gamma^2+\Gamma Vb\left(1-\frac{B}{(B^2+b^2)^{1/2}}\right)+\frac{b^2V^2}{8\surd 2}\right\}, \tag{2.5d}$$

$$V = -4\Gamma b/(B^2+b^2), \quad (B^2)_t = -2\alpha bW, \tag{2.5e,f}$$

where the prime denotes differentiation with respect to z. In this integration process the following simplifying assumptions were made.

(i) A *polytropic model* of the atmosphere was used, $p_a = C\rho_a^{k\gamma}$, $C = p_0/\rho_0^{k\gamma}$, with $\gamma = c_p/c_v$, the ratio of the specific heats at constant pressure and constant volume, respectively, and index of polytropy, k.

(ii) The velocities v_1 and v_2 obey the order of magnitude inequalities

$$(v_2)_t \ll (v_1)_t, \quad v_z \ll v_r$$

so that the azimuthal momentum balance can be split into two equations according to these orders, see the Appendix. The splitting is shown in (2.5*e*) and (2.5*f*).

We also note that (2.5*d*) differs from the corresponding equation derived by Carrier *et al.* (1985) as explained in the Appendix.

Let us introduce the functions

$$F(z,t) = \rho_a b^2 W, \; G(z,t) = \rho_a b^2 W^2, \; H(z,t) = c_p T_a b^2 f W,$$

describing axial fluxes of mass, axial momentum and energy along the vertical line above the source. Then the set (2.5) can be rewritten in the following form:

$$F' = 2\alpha(\rho_a G)^{1/2}, \quad G' = 2\left(\frac{gHF}{c_p T_a G} + P'\right), \quad H' = -\frac{2g(1-k)}{k(\gamma-1)}F, \qquad (2.6a\text{–}c)$$

$$P \equiv \sigma b^2 = 2\rho_a\left\{(2-\surd 2)\frac{F\Gamma^2}{(\rho_a G)^{1/2} B} + \frac{F\Gamma V}{(\rho_a G)^{1/2}} \times\left(1 - \frac{B}{[B^2 + F^2/(\rho_a G)]^{1/2}}\right) + \frac{F^2 V^2}{8\surd 2 \rho_a G}\right\}, \qquad (2.6d)$$

$$V = -\frac{4\Gamma F}{(\rho_a G)^{1/2}(B^2 + F^2/(\rho_a G))}, \quad (B^2)_t = -2\alpha(G/\rho_a)^{1/2}. \qquad (2.7a, b)$$

Functions W, b, f are related to the fluxes F, G, H by the expressions

$$W = G/F, \quad b = F/(\rho_a G)^{1/2}, \quad f = \rho_a H/(c_p T_a F).$$

If a background vortex is absent ($\Gamma = 0$), the pressure deficit σ is zero, and we have from (2.6)

$$F' = 2\alpha(\rho_a G)^{1/2}, \quad G' = \frac{2gHF}{c_p T_a G}, \quad H' = -\frac{2g(1-k)F}{k(\gamma-1)}. \qquad (2.8a\text{–}c)$$

The set (2.8) describes a steady non-swirling plume, since in the model considered the plume rotation is initiated by an external vortex only.

In case of an adiabatic atmosphere ($k = 1$, $H = \text{const}$) there follows

$$F' = 2\alpha(\rho_a G)^{1/2}, \quad G' = \frac{2gHF}{c_p T_a G}. \qquad (2.9)$$

These equations have a similarity solution (Carrier *et al.* 1985) if the mass density ρ_a and pressure p_a are independent of the height z. In fact, elimination of the function F yields

$$(GG')' = CG^{1/2},$$

where the constant $C = 4\alpha\rho_a^{1/2} gH/(c_p T_a)$. If the solution is sought in the form $G = A(z+c)^n$, we obtain $n = 4/3$ and $A = (9C/20)^{2/3}$. Therefore, the similarity solution of (2.9) is

$$F = \tfrac{6}{5}\alpha\rho_a\left(\frac{18\alpha g\epsilon}{5\pi}\right)^{1/3}(z+c)^{5/3}, \quad G = \rho_a\left(\frac{18\alpha g\epsilon}{5\pi}\right)^{2/3}(z+c)^{4/3}, \qquad (2.10a, b)$$

where
$$\epsilon = \pi(\gamma-1)H/(2\gamma p_a).$$

For the primitive functions we have

$$b = \frac{6\alpha}{5}(z+c), \quad W = \frac{5}{6\alpha}\left(\frac{18\alpha g\epsilon}{5\pi(z+c)}\right)^{1/3}, \quad f = \left(\frac{5\rho_a\epsilon}{3\alpha\pi}\right)\left(\frac{5\pi}{18\alpha g\epsilon}\right)^{1/3}(z+c)^{-5/3}.$$

The general case of a stratified, two-layer polytropic atmosphere

$$k = \begin{cases} k_2, & 0 \leqslant z \leqslant z_1, \\ k_2, & z_1 \leqslant z \leqslant \infty, \end{cases} \quad (1/\gamma \leqslant k_{1,2} \leqslant 1)$$

described by (2.6) has no analytical solution, and it is necessary to solve these equations numerically.

2.2. *Numerical algorithm*

We describe now the general scheme for the numerical calculations of (2.6), (2.7). Continuous independent variables and dependent functions are replaced by discrete pointwise analogues: $z_j = jh$, $t^n = n\tau$,

$$Q(z_j, t^n) = Q_j^n; \quad j = 0, 1, \ldots, J; \quad n = 0, 1, \ldots, N; \quad J = z_{max}/h, \quad N = t_{max}/\tau;$$

h and τ are spatial and temporal mesh sizes and $Q = (F, G, H, B)$. Assuming that some initial profile for the background vortex $B(z, 0) = B^0(z)$ and boundary conditions $q(0, t) = q_0$, where $q = (F, G, H)$, are prescribed, we obtain the pressure deficit σ at $z = 0$ and $t = 0$. Thereafter, the ordinary differential equations (2.6*a*–*c*) are solved by the fourth-order Runge–Kutta method; they determine the functions q_j^0 for all indices j. These functions correspond to the initial 'radius' of the background vortex. Substituting q_i^0 into (2.7) and stepping forward in time yields values of B_j^1. Making use of these values, we solve (2.6) etc. until $t = t_{max}$.

It is now useful to rewrite (2.6), (2.7) in dimensionless variables, by introducing scaled values for length, time and velocity as follows:

$$\bar{z} = g_0^{-1/5}\epsilon^{2/5}, \quad \bar{t} = g_0^{-3/5}\epsilon^{1/5}, \quad \bar{u} = g_0^{2/5}\epsilon^{1/5}, \tag{2.11 a–c}$$

where
$$g_0 = [\rho_a^{\gamma-1}(p_a/\rho_a^{\gamma})']_{z=0}$$

is the typical value of acceleration in the non-adiabatic atmosphere at the ground level. For the polytropic atmosphere which we consider, $g_0 = g(1-k_1)/k_1$. Scaled values for the functions F, G, H are

$$\bar{F} = \rho_0\bar{z}^2\bar{u}, \quad \bar{G} = \rho_0\bar{z}^2\bar{u}^2, \quad \bar{H} = c_p T_0\rho_0\bar{u}\bar{z}^2. \tag{2.12 a–c}$$

In the dimensionless variables (2.11), equations (2.6), (2.7) take the following form:

$$F' = 2\alpha(\rho_a G)^{1/2}, \quad G' = 2\left(\frac{gHF}{g_0 T_a G} + P'\right), \quad H' = -\frac{2(k-1)g\rho_0\bar{z}}{\gamma k p_0}F, \tag{2.13 a–c}$$

$$(B^2)t = -2\alpha(G/\rho_a)^{1/2}. \tag{2.14}$$

Here ρ_a is the ambient mass density scaled by ρ_a, T_a is the ambient temperature scaled by T_0 and

$$P = 2\rho_a F\Gamma^2\{(2-\surd 2)(\rho_a G)^{-1/2}B^{-1} + \surd 2 F^3X^{-2} - 4FX^{-1}(1 - B(\rho_a G)^{1/2}X^{-1/2}\},$$
$$X = F^2 + \rho_a GB^2.$$

When solving (2.13) numerically, some difficulty arises because of their non-standard form due to the term P'. However, if we represent P' as

$$P' = \sum_{i=1}^{5}\frac{\partial P}{\partial U_i}U_i', \quad U = (F, G, B, \rho_a, \Gamma),$$

then it is possible to rewrite the set (2.13) in the standard form

$$F' = 2\alpha(\rho_a G)^{1/2}, \tag{2.15a}$$

$$G' = 2\left(1-2\frac{\partial P}{\partial G}\right)^{-1}\left\{\frac{gHF}{g_0 T_a G}+2\alpha\frac{\partial P}{\partial F}(\rho_a G)^{1/2}+\frac{\partial P}{\partial B}B'+\frac{\partial P}{\partial \rho_a}\rho_a'+\frac{\partial P}{\partial \Gamma}\Gamma'\right\} \tag{2.15b}$$

$$H' = -\frac{2(1-k)g\rho_0 \bar{z}}{\gamma k p_0}F. \tag{2.15c}$$

The functions F, G, H (and B, W, f) depend parametrically on time. As initial conditions for (2.15) we take the similarity solution (2.10) at $z = 0$, corresponding to the non-rotating plume in an adiabatic atmosphere, e.g.

$$F_0 = \left(\frac{5g}{2\alpha\pi g_0}\right)^{1/3} b_0^{5/3}, \quad G_0 = \left(\frac{5g}{2\alpha\pi g_0}\right)^{2/3} b_0^{4/3}, \quad H_0 = \frac{2}{\pi}. \tag{2.16a–c}$$

Therefore, the problem of vortex flow above a horizontal point source of mass, momentum and heat reduces to the numerical solution of (2.14), (2.15) subject to an initial condition B_j^0 (sufficiently arbitrary) and boundary conditions (2.16).

2.3. *Results and discussion*

First, note that the plume scale b_0 and the vertical velocity W_0 on the plume axis at the ground level $z = 0$ are constants (because of boundary conditions), and hence owing to (2.5f) the typical size ('radius') of the external background vortex is

$$B_0(t) - (B_0^2(0) - 2\alpha b_0 W_0 t)^{1/2}. \tag{2.17}$$

This value equals the 'radius' b_0 at time

$$t_* = \frac{B_0^2(0)-b_0^2}{2\alpha b_0 W_0}. \tag{2.18}$$

The rotation velocity of the whole structure is

$$v(r,z,t) = \frac{\Gamma(z)}{r}\left\{1-\exp\left(-\frac{r^2}{B^2(z,t)}\right)-\frac{4r^2}{B^2(z,t)+b^2(z,t)}\exp\left(-\frac{r^2}{b^2(z,t)}\right)\right\} \tag{2.19}$$

and near the Earth's surface

$$v(r,0,t) = \frac{\Gamma(0)}{r}\left\{1-\exp\left(-\frac{r^2}{B_0^2(t)}\right)-\frac{4r^2}{B_0^2(t)+b_0^2}\exp\left(-\frac{r^2}{b_0^2}\right)\right\}. \tag{2.20}$$

For the calculations we choose the following values of parameters:

$$\rho_0 = 1.16\ \mathrm{kg\,m^{-3}}, \quad p_0 = 10^5\ \mathrm{kg\,m^{-1}\,s^{-2}}, \quad \gamma = 1.4,$$
$$k_1 = 0.99, \quad k_2 = 0.83, \quad z_1 = 2.74\ \mathrm{km}.$$

With these and $\epsilon = 2.6\times 10^6\ \mathrm{m^3\,s^{-1}}$ the typical acceleration g_0 and scale values become

$$g_0 = 9.9\times 10^{-2}\ \mathrm{m\,s^{-1}}, \quad z = 584\ \mathrm{m}, \quad t = 76.8\ \mathrm{s}, \quad \bar{u} = 7.61\ \mathrm{m\,s^{-1}}.$$

The external vortex charateristics were taken as suggested by Carrier *et al.* (1985): $\Gamma(0) = 3.2\times 10^4\ \mathrm{m^2\,s^{-1}}$ with the circulation $\Gamma(z)$, decreasing linearly in z up to $z = 3$ km, beyond which the value of Γ is zero; similarly, $B^0(0) = 8$ km, and it increases weakly with height:

$$\Gamma(z) = \begin{cases} 3.2\times 10^4(1-z\ \mathrm{m}/3000)\ \mathrm{m^2\,s^{-1}}, & z \leqslant 3000\ \mathrm{m}, \\ 0 & z \geqslant 3000\ \mathrm{m}, \end{cases}$$

$$B^0(0) = (8\times 10^3 + 0.49z\ \mathrm{m})\ \mathrm{m}.$$

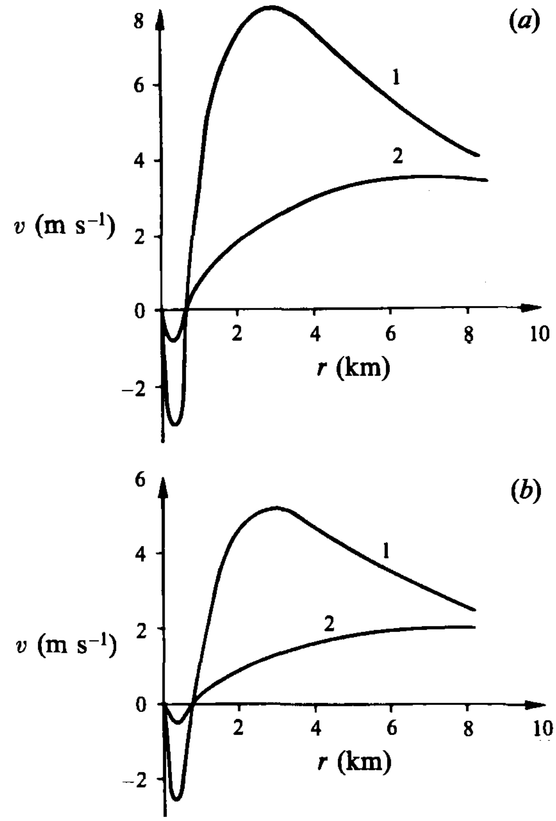

FIGURE 1. Azimuthal velocity v m s^{-1} *vs*. radial distance from the plume axis r km at times $t = 64$ min (curve 1) and $t = 128$ min (curve 2) after onset of motion. Source strength $\epsilon = 2.6 \times 10^6$ m^3 s^{-1}. (*a*) At ground level $z = 0$ km; (*b*) at $z = 1.18$ km above the ground.

For these values the initial azimuthal velocity at ground level $z = 0$ and at $r = B^0(0)$, the 'outer edge' of the background vortex, equals 2.5 m s^{-1}, for the plume size we choose 584 m (or in dimensionless form $b_0 = 1$). The above choice of parameter values corresponds to the scales appropriate to the Hamburg firestorm and is thus relevant for real-world situations, see Carrier *et al.* (1985). Figures 1–3 summarize the results of the numerical integration of (2.15), (2.16) as they were obtained with the above parameters. Figures 1(*a*) and 1(*b*) display the radial distributions of the azimuthal velocity v at times $t = 64$ and 128 min after onset of the external vortex motion at ground level $z = 0$ (figure 1*a*) and $z = 1.18$ km (figure 1*b*), respectively. The rotation distant from the plume axis is cyclonic in consonance with the driving vortex, but close to the plume axis it is antiyclonic. This can be explained as follows. According to (2.19), as $r \to 0$, i.e. $r \ll B$

$$v(r,z,t) = \frac{r\Gamma}{B^2}\left[1 - \frac{4}{1+(b/B)^2}\exp\left(-\frac{r^2}{b^2}\right)\right], \tag{2.21}$$

and in the early phase of plume rotation when $b \ll B$

$$v(r,z,t) = \frac{r\Gamma}{B^2}[1 - 4\exp(-r^2/b^2)], \quad r \to 0, \tag{2.22}$$

and if also $r \ll b$

$$v(r,z,t) = -3r\Gamma/B.$$

	Time		Radius r_*, r_{**}		r_{**} $v(r_*, 0, t)$	
	Dimensionless	min.	Dimensionless	km	Dimensionless	m s^{-1}
(*a*)	25	32	—	—	0.05	0.40
	50	64	—	—	0.07	0.56
	100	128	—	—	0.40	3.0
(*b*)	25	32	13.7	8	0.40	3.0
	50	64	12,0	7	0.46	3.5
	100	128	5,0	2,9	1.06	8.1

TABLE 1. Maximum ground level modulus of the rotation velocity in the anticyclonic core of the plume arising at (*a*) $r = r_*$ and (*b*) in the outer, cyclonic part arising at $r = r_{**}$

Near the ground ($z = 0$) at $r \ll B$, $b \approx b_0 \ll B$ the velocity equals

$$v(r, 0, t) = \frac{r\Gamma(0)}{B_0^2(t)}[1 - 4\exp(-r^2/b_0^2)], \quad r \to 0, \tag{2.23}$$

from which we infer that the rotation velocity $v(r, 0, t), t \to 0$, vanishes at $r = r_1 = (\ln 4)^{1/2} b_0 = 1.18$ (or 689 m in dimensional form). In the interval $0 \leqslant r \leqslant r_1$ the maximum modulus of the rotation velocity is

$$|v(r_*, 0, t)| = \frac{2r_*^3\,\Gamma(0)}{(b_0^2 - 2r_*^2)\,B_0^2(t)} \quad \text{for} \quad r \to 0, \tag{2.24}$$

and it arises where

$$r_*^2 = b_0^2 \ln(4(1 - 2r_*^2/(b_0^2))), \tag{2.25}$$

from which we deduce $r_* = 0.57b_0$ (i.e. 333 m) and $|v(r_*, 0, t)| = 7.97/B_0^2(t)$, valid for small time. This value increases with time since the external vortex size decreases. In particular, we obtain the values of table 1(*a*). At the plume 'edge' $r = b$ and for $b \ll B$

$$v(b, z, t) = -(\Gamma b/B^2)(4/e - 1) \quad \text{for} \quad t \to 0. \tag{2.26}$$

The turning radius from anticyclonic to cyclonic rotation is $r = r_1$; beyond it, the rotation velocity increases and reaches a relative maximum at $r = r_{**}$ and falls off for $r > r_{**}$. A selection of these maxima is given in table 1(*b*). Evidently, the size of the compound plume, i.e. the united structure of the plume 'core' and the external vortex, decreases with time. Obviously, as $r \to \infty, v(r, z, t) \sim \Gamma/r$.

At height $z \neq 0$ the clockwise rotating core is separated from the counterclockwise rotating outer part at the radius:

$$r_1 = (\ln 4)^{1/2} b(z) = 1.18b(z) \tag{2.27}$$

and

$$r_*^2 = b^2(z) \ln[4(1 - 2r_*^2(z))/b^2(z)]. \tag{2.28}$$

For conditions appropriate for figure 1(b), z = 1.18 km above ground

$$r_1 = 1.5 \quad (867\text{ m}), \quad r_* = 0.72 \quad (420\text{ m}).$$

The existence of an interior region with a weak rotation in the opposite direction to the greater part of the compound rotating structure is a phenomenon referred to a 'typhoon's eye', and is obviously manifest here too.

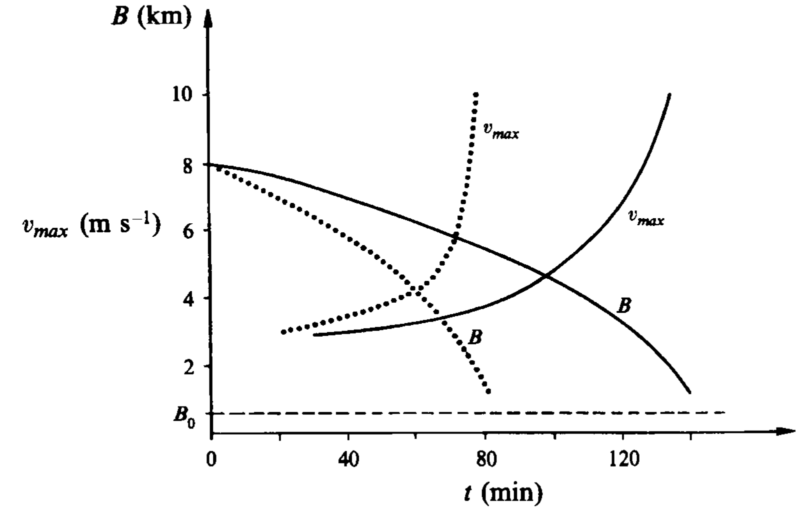

FIGURE 2. Time dependence of the external vortex size B and maximum rotation velocity v_{max} at ground level. Solid lines correspond to the source power $\epsilon = 2.6 \times 10^6$ m^3 s^{-1}, dotted lines to $\epsilon = 11.3 \times 10^6$ m^3 s^{-1}.

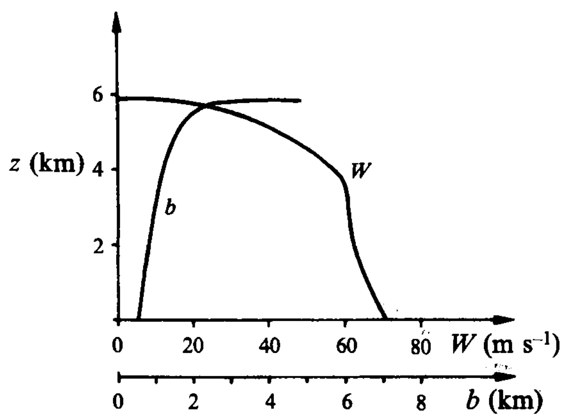

FIGURE 3. Dependence of the plume size b and vertical velocity W on height at $\epsilon = 2.6 \times 10^6$ m^3 s^{-1}.

Figure 2 displays the temporal distributions of the size of the external vortex, B_0, and rotation velocity v_{max} at ground level ($z = 0$) for two values of the source strength. As follows from formula (2.18) and is confirmed computationally, the external vortex contracts down to the plume size b at $t = 142$ min; however, the theory cannot apply when $B \sim b$ and so is limited to early time behaviour.

In figure 3 the distributions of the plume radius $b(z)$ and axial velocity $W(z)$ at $t = 64$ min are presented. The time dependence of these functions is weak since the pressure deficit σ (or P) corresponding to the parameters chosen (as in Carrier *et al.* 1985) is small. The plume rises to the terminal height $z_{max} = 5.8$ km where the vertical velocity vanishes and the plume spreads out horizontally.

It is interesting to analyse the response of the compound plume to variations in the driving sources. When the external vortex circulation is increased by a factor of two, then the results are as follows: the plume size $b(z)$ increases by about 5% and the vertical velocity on the plume axis is larger by 6–10% (depending on height). This increase can be explained by larger contributions from the pressure gradient P'. The azimuthal velocity is up to at least twice as large, which is plausible.

Next, we increase the source strength to the value $\epsilon = 11.3 \times 10^6\ \mathrm{m^3\ s^{-1}}$, but retain the initial characteristics of the external vortex $B_0(0) = 8$ km, $\Gamma(0) = 3.2 \times 10^4\ \mathrm{m\ s^{-1}}$ and the source size $b_0 = 584$ m. These imply the length, time and velocity scales $\bar{z} = 1052$ m, $\bar{t} = 1035$ s, $\bar{u} = 10.2\ \mathrm{m\ s^{-1}}$. In figure 2 the time dependence of the external vortex size and maximum velocity of the rotation of the compound structure near the ground level is shown by dotted lines. The collapse of the external vortex occurs more rapidly than in the case of the smaller power source, and the typical time of the collapse equals $t_* = 83$ min. The azimuthal velocity $v(r, 0, t)$ changes its direction of rotation at $r_1 = b_0(\ln 4)^{1/2} = 0.655$ (for the given power we have $b_0 = 0.555$). In dimensional variables the radius $r_1 = 689$ m equals the corresponding value for the case of $\epsilon = 2.6 \times 10^6\ \mathrm{m^3\ s^{-1}}$. From (2.24), (2.25) we have $r_* = 0.316$ (333 m), $v(r_*, 0, t) = 1.81/B_0^2(t)$. The fire plume rises to the height $z_{max} = 7.8$ (8.2 km) which is obviously higher than for the weaker source strength.

Finally, since our equations (2.5) differ from those of Carrier *et al.* (1985) in the expression for the pressure deficit P, it would be interesting to quantify this difference. For the above parameter choice this difference is very small (because P is small) but when P is larger at larger values of the circulation Γ (note $P \sim \Gamma^2$) the difference between our expression and Carrier *et al.*'s for P will be significant, but need not be quantified, because the expression for the pressure deficit by Carrier *et al.* (1985) is simply wrong and we correct it here.

3. Steady rotating plume supported by small-scale helical turbulence

We now suppose that there is no externally applied circulation field but instead a helical turbulent field with axial symmetry concentric with the fire source. Such a plume can have a rotational velocity component. Helical turbulence is a small-scale feature providing the background for the development of the inverse energy cascade and large-scale structure (Levich, Shtilman & Tur 1991; Berezin *et al.* 1991). At large Reynolds numbers, which we assume, a change of this background does not influence the large-scale self-organization and thus is equally ineffective in its dissipative scales. Helical turbulence is invariant under translations and rotations, but not under mirror reflections, i.e. helical turbulence lacks parity-invariant symmetry. For details, see Berezin *et al.* (1991). Consider such a situation.

3.1. *Formulation of the problem*

The governing equations are (2.3) in which terms due to the helicity are added and unsteady time-dependent contributions are omitted. With the help of (3.10) of Berezin *et al.* (1991) the equations can be complemented by the contributions from steady axisymmetric turbulence and take the form

$$(r\rho_a u)_r + (r\rho_a w)_z = 0, \tag{3.1a}$$

$$(r\rho_a u^2)_r + (r\rho_a uw)_z + rp_r - \rho_a v^2 - \Omega(r\rho_a Sv)_z = 0, \tag{3.1b}$$

$$(r^2\rho_a uv)_r + (r^2\rho_a wv)_z + \Omega(r^2\rho_a Su)_z = 0, \tag{3.1c}$$

$$(r\rho_a uw)_r + (r\rho_a w^2)_z + rp_z + r\rho g + \Omega(r\rho_a Sv)_r = 0, \tag{3.1d}$$

$$(r\rho_a uT)_r + (r\rho_a wT)_z + r\rho_a gw/c_p = 0, \tag{3.1e}$$

$$p_a = \bar{R}\rho T, \quad (p_a)_z = -g\rho_a,$$

where $S = S(r, z)$ is the spatially dependent helicity coefficient. Evidently, the effect of helicity enters all three components of the momentum balance. We represent the solutions of (3.1) as Gaussian profiles similar to (2.4), i.e.

$$w(r,z) = W(z)\,\mathrm{e}^{\psi}, \tag{3.2a}$$

$$p_a(z)-p(r,z,t) = \sigma(z)\,\mathrm{e}^{\psi}, \quad \rho_a(z)-\rho(r,z,t) = \rho_a(z)f(z)\,\mathrm{e}^{\psi}, \tag{3.2b, c}$$

$$v(r,z) = \frac{rV(z)}{b(z)}\,\mathrm{e}^{\psi}, \quad \psi(r,z) = -\frac{r^2}{b^2(z)} \tag{3.2d, e}$$

and assume that the flow excites a helical turbulent field as follows:

$$S(r,z) = S_0 \exp(-r^2/a^2). \tag{3.3}$$

Here S_0 and a are constants characterizing the value of the helicity on the plume vertical axis and the size of the helicity domain, respectively. Substituting (3.2) into (3.1) and integrating them over r from $r = 0$ to $r \to \infty$, yields the following ordinary differential equations (in the same notation and same dimensionless variables as in §2):

$$F' = 2\alpha(\rho_a G)^{1/2}, \quad G' = 2\left(\frac{gHF}{g_0 T_a G}+P'\right), \quad H' = -\frac{2g(1-k)\rho_0 \bar{z}}{k\gamma p_0}F, \tag{3.4a–c}$$

$$V = 2\beta a^4 F^{-2}(\rho_a G)^{1/2}\left(\frac{\rho_a GF}{F^2+\rho_a a^2 G}\right)', \tag{3.4d}$$

$$P = \frac{F^2V^2}{4\sqrt{2}G}+\frac{\beta a^3 F}{2(\rho_a G)^{1/2}}\left(\frac{\rho_a(\rho_a G)^{1/2}\,VF^2}{(F^2+\rho_a a^2 G)^{3/2}}\right)', \tag{3.4e}$$

where $\beta = \Omega S_0/\bar{u}$ is the dimensionless helicity parameter. Equation (3.4d) leads to the possibility of the existence of steady rotation in the presence of small-scale helical turbulence. The azimuthal velocity v and the plume width b are related to the functions F, G, V by the formulas

$$v(r,z) = \frac{V(z)}{b(z)}r\exp\left(-\frac{r^2}{b^2(z)}\right), \quad b(z) = F/(\rho_a G)^{1/2}.$$

At any fixed z the azimuthal velocity possesses a maximum v_{max} which arises at $r_*(z) = b(z)/\sqrt{2}$. This maximum value equals

$$v_{max}(z) = \frac{V(z)}{b(z)\,e^{1/2}}.$$

3.2. *Special case. Adiabatic atmosphere and small helicity*

In this case the azimuthal velocity 'amplitude' $V(z)$ is of first order and the pressure deficit P of second order in the helicity parameter β, which is small. Solutions of (3.4) are therefore sought in the form of asymptotic expansions of the following type:

$$\phi = \phi^{(0)}+\beta\phi^{(1)}+\ldots, \quad \phi = (F,G,H), \quad V = \beta V^{(1)}+\ldots, \quad P = \beta^2 P^{(2)}+\ldots. \tag{3.5a–d}$$

The leading-order functions are solutions of the following equations describing a non-rotating plume:

$$F^{(0)\prime} = 2\alpha(\rho_a G^{(0)})^{1/2}, \quad G^{(0)\prime} = \frac{2gH^{(0)}F^{(0)}}{g_0 T_a G^{(0)}}, \quad H^{(0)\prime} = -2(1-k)g\rho_0 \bar{z}\frac{F^{(0)}}{k\gamma p_0}. \tag{3.6a–c}$$

This set is the dimensionless analogue of (2.6). In case of an adiabatic atmosphere with the additional assumption that the pressure and mass density are independent of the height z, (3.6) possess the similarity solution (in dimensionless form)

$$F^{(0)} = \tfrac{6}{5}\alpha\rho_a\left(\frac{18\alpha g}{5\pi g_0}\right)^{1/3}(z+c)^{5/3}, \quad G^{(0)} = \rho_a\left(\frac{18\alpha g}{5\pi g_0}\right)^{2/3}(z+c)^{4/3}, \quad H^{(0)} = \frac{2}{\pi}. \tag{3.7 a–c}$$

By substituting the functions (3.7) into (3.4d), one finds the 'amplitude' of the azimuthal velocity in the linear approximation,

$$V(\zeta) = \frac{4\alpha\beta a^4(a^2-\zeta b_0^2/5)}{\zeta b_0^2(a^2+\zeta b_0^2)^2} + O(\beta^2), \quad \zeta = \left(1+\frac{6\alpha z}{5b_0}\right)^2, \tag{3.8}$$

where b_0 is a typical value of the plume size on the ground. The azimuthal velocity in the linear approximation becomes

$$v(r,z) = \frac{4\alpha\beta a^4(a^2-\zeta b_0^2/5)}{\zeta^{3/2} b_0^3(a^2+\zeta b_0^2)^2}\, r\exp\left(-\frac{r^2}{b^2(z)}\right), \quad b^2 = \zeta b_0^2, \tag{3.9}$$

which for fixed height z is of one sign (positive or negative) and vanishes at $r = 0$ and as $r \to \infty$. At the ground level $\zeta = 1$ ($z = 0$) it takes the form

$$v(r,0) = \frac{4\alpha\beta a^4(a^2-a_*^2)}{b_0^3(a^2+b_0^2)^2}\, r\exp\left(-\frac{r^2}{b_0^2}\right), \quad a_* = \frac{b_0}{\sqrt{5}} = 0.45 b_0.$$

Clearly, for $a > a_*(a < a_*), v(r,0)$ is positive (negative), and for $a = a_*$ it vanishes. Recall that the quantity a defines the radius of influence of small-scale turbulence. Also, for finite r and as $z \to \infty$,

$$v(r,z) \to -\frac{625\beta a^4}{1944\alpha^4} r z^{-5}.$$

If the dimension of the helical-turbulent domain is less than a_* the vortex azimuthal velocity is negative at all heights, and has the following maximum modulus:

$$\max_r |v(r,\zeta)| = \frac{4\alpha\beta a^4(\zeta b_0^2/5 - a^2)}{(2e)^{1/2}\,\zeta b_0^2(a^2+\zeta b_0^2)^2} \quad \text{at} \quad r_* = \frac{b(z)}{\sqrt{2}}. \tag{3.10}$$

If the maximum of $|v(r,z)|$ is sought over $r \in (0,\infty)$ and $z \in (0,\infty)$ then one obtains

$$\max_{r,z} |v(r,\zeta)| \equiv v_{max} = \frac{64\alpha\beta(\sqrt{(265)}-5)}{5(15+\sqrt{265})(19+\sqrt{265})^2(2e)^{1/2}} = 2.25\times 10^{-3}\alpha\beta \tag{3.11}$$

$$\text{at} \quad r_* = \frac{b(z^*)}{\sqrt{2}}, \quad z_* = \frac{5}{6}\frac{1}{\alpha}\left(\tfrac{1}{2}a(15+\sqrt{265})^{1/2} - b_0\right).$$

If the dimension of the helical-turbulent domain is larger than a_*, the azimuthal velocity is positive at $z = 0$ and decreases with height, becomes zero at $z \equiv z_1 = (5/(6\alpha))(a\sqrt{5}-b_0) < z_*$ and remains negative for $z > z_1$, decreasing in modulus like z^{-5} as $z \to \infty$. At the height $z = z_1$ the vortex changes the direction of its rotation.

Consider the special case when the dimension a of the turbulent domain equals the plume size on the ground, i.e $a = b_0$. Then the vortex azimuthal velocity is

$$v(r,\zeta) = \frac{4\alpha\beta(1-\zeta/5)}{b_0\zeta^{3/2}(1+\zeta)^2} r\exp\left(-\frac{r^2}{b_0^2\zeta}\right).$$

At the ground level ($z = 0$ or $\zeta = 1$) this becomes

$$v(r) = \frac{4\alpha\beta}{5b_0} r\exp\left(-\frac{r^2}{b_0^2}\right);$$

the vortex changes the direction of its rotation at

$$z_1 = \frac{5b_0}{6\alpha}(\sqrt{5}-1) \approx \frac{b_0}{\alpha},$$

and the maximum of the modulus of the azimuthal velocity is given by (3.11) and arises at

$$z_* = \frac{5b_0}{6\alpha}(\tfrac{1}{2}(15+\sqrt{265})^{1/2}-1)).$$

The full nonlinear problem as well as the case of a real stratified (polytropic) atmosphere with large helicity requires numerical solution of (3.4).

3.3. *Polytropic atmosphere with O(1)-helicity*

For the polytropic atmosphere considered in §2 we have

$$\rho_a' = -g\rho_0^{k\gamma}\rho_a^{2-k\gamma}/(p_0 k\gamma).$$

Hence the typical value of the acceleration in such an atmosphere on the ground is

$$g_0 = g(1-k_1)/k_1,$$

where $k_1 = k|_{z=0}$.

Equations (3.4) can be solved by iteration using the following scheme:

(i) determine the functions F, G, H for the non-rotating plume from the equations:

$$F' = 2\alpha(\rho_a G)^{1/2}, \quad G' = \frac{2k_1\rho_a HF}{(1-k_1)p_a G}, \quad H' = -\frac{2(1-k)\rho_0 g\bar{z}F}{k\gamma p_0};$$

(ii) substitute the solutions into the expression for the amplitude of the azimuthal velocity V, (3.4*d*);

(iii) calculate the pressure deficit P, (3.4*c*);

(iv) substituting this expression for P into the right-hand side of (3.4*b*) yields

$$G' = 2\left[\left(\frac{k_1\rho_a HF}{(1-k_1)p_a G}\right)+P'\right],$$

This process yields a 'new' updated solution for the function G;

(v) determine 'new' updated values of the functions F, H from (3.4*a*) and (3.4*c*) etc.

This scheme has led to a fairly rapid convergence.

Let us present some results of the numerical computations that were performed with the following model parameters: $\epsilon = 2.6\times10^6$ m^3 s^{-1}, $\beta = 0.5$, $a = b_0 = (584$ m$)$; the

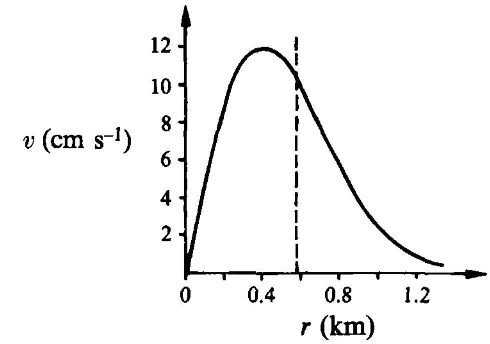

FIGURE 4. Radial distribution of the rotation velocity at ground level. The dotted line denotes the radius of the helical turbulent domain.

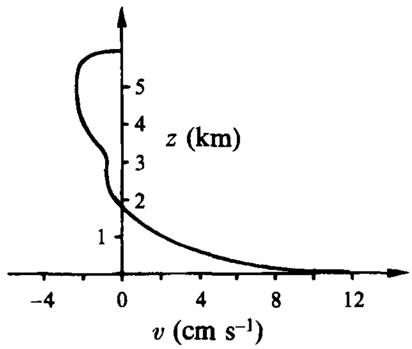

FIGURE 5. Dependence of the rotation velocity v_{max} on height.

other parameters are the same as in §2. Figures 4 and 5 show the radial distribution of the rotation velocity of the fire plume considered at the ground level and the distribution of the maximum rotation velocity *vs.* height z, respectively. This maximum azimuthal velocity occurs at the ground level at $r = 0.7$ (410 m) and equals 11.9 cm s^{-1}. The chosen value of the parameter $a = b_0$, associated with a size of the developed helical turbulence, is larger than the characteristic value of $a_* = 0.45$. Therefore, as was the case for the adiabatic atmosphere, the direction of the convective column rotation changes with height: the fire plume rotates counterclockwise for $z < z_1 = 1.46$ km; at $z > z_1$ the rotation becomes clockwise, approaching the maximum velocity of 2 cm s^{-1} at the height $z = 4.67$ km, and then becoming zero at $z = 5.95$ km, where the plume spreads out horizontally. The relatively abrupt change of the form of the curve $v_{max}(z)$ at $z \approx 3$ km corresponds to the changes of the polytropic index in the two-layer atmosphere considered. The most intensive rotation of the convective column occurs near the Earth's surface.

4. Concluding remarks

We have considered two mechanisms of rotation of fire plumes initiated by axisymmetric sources of mass, momentum, and heat. The first mechanism is associated with the postulated presence of an external vortex, which interacts with the fire plume and creates a rotating structure (Carrier *et al.* 1985). The second mechanism (proposed by the authors) is based on the assumption that in a zone inside the fire plume there exists a domain with small-scale helical turbulence. The balance laws of mass, momentum and energy in the non-dissipative form are integrated over the plume area: by assuming similarity profiles for the variables involved, reduced equations are derived for their values on the axis. The emerging nonlinear differential equations can, in simple situations, be studied analytically and solved numerically in more realistic situations with reasonable effort.

The helical turbulent mechanism gives rise to the possibility for fire plumes to rotate. It is shown that the convective column rotates either unidirectionally, or the rotation changes its direction at a certain height. Such cases depend on a relation between the size of the helical turbulent domain and the plume radius at the ground level (which are both input parameters). Finally, we note that the theory can be extended by taking into account a turbulent viscosity, associated with the symmetrical part of the stress tensor, and the volume heat sources.

Appendix

The derivation of (2.5) is based on the substitution of the functions (2.4) into (2.3), integrating them over the radial distance from $r = 0$ to $r \to \infty$, and making use of Taylor's entrainment assumption.

(i) From the continuity equation (2.3*a*) we have

$$\rho_a ru|_\infty^0 + (\rho_a W\phi_1)' = 0 \quad \text{or} \quad (\rho_a W\phi_1)' = \alpha\rho_a bW,$$

where $\phi_1 = \int_0^\infty r\exp(-r^2/b^2)\,dr$. Substituting $\phi_1 = b^2/2$ yields (2.5*a*).

(ii) Subtracting the hydrostatic equation (2.2) from the equation of axial momentum (2.3*d*) yields

$$(r\rho_a uw)_r + (r\rho_a W^2)_z = r(p_a - p)_z + rg(\rho_a - \rho).$$

Substituting (2.4) transforms this to

$$\rho_a Wru\exp(-r^2/b^2)|_0^\infty + (\rho_a W^2\phi_2)' = (\sigma\phi_1)' + gf\phi_1,$$

where $\phi_2 = \int_0^\infty r\exp(-2r^2/b^2)\,dr$. The first term equals zero. Substituting $\phi_2 = b^2/4$, $\phi_1 = b^2/2$ yields (2.5*b*).

(iii) The balance law of energy (2.3*e*) can be rewritten in the following form:

$$(r\rho_a u(T-T_a))_r + (r\rho_a w(T-T_a))_z + r((T_a)_z + g/c_p)\rho_a w = 0.$$

Making use of (2.3*f*) yields

$$((ruT_a(\rho_a-\rho))_r + (rwT_a(\rho_a-\rho))_z + r((T_a)_z + g/c_p)\rho_a w = 0$$

and upon use of (2.4*c*)

$$fT_a(ru)\exp(-r^2/b^2)|_0^\infty + (fT_a W\phi_2)' + ((T_a)_z + g/c_p)\rho_a W\phi_1 = 0. \qquad \text{(A 1)}$$

We consider a two-layer polytropic model for a dry stably stratified atmosphere such that $p_a = C\rho_a^{k\gamma}$, with index $k = k_1$ within $0 \leqslant z \leqslant z_1$ and $k = k_2$ for $z > z_1$, where z_1 is some given height, $\gamma^{-1} \leqslant k_{1,2} \leqslant 1$, γ being the ratio of the specific heats for air, with $C = p_0/\rho_0^{k\gamma}$, the subscript zero denoting a constant ground level value. For such an atmosphere

$$T_a = \frac{C}{R}\rho_a^{k\gamma-1}, \quad (T_a)_z = \frac{C}{R}(k\gamma-1)\rho_a^{k\gamma-2}(\rho_a)_z, \qquad \text{(A 2}a, b\text{)}$$

$$(\rho_a)_z = -\frac{g}{Ck\gamma}\rho_a^{2-k\gamma}, \quad (T_a)_z = -\frac{g(k\gamma-1)}{c_p k(\gamma-1)}, \quad (T_a)_z + \frac{g}{c_p} = \frac{g(1-k)}{c_p k(\gamma-1)}. \qquad \text{(A 2}c\text{–}e\text{)}$$

Substituting (A 2) into (A 1) yields (25*c*).

(iv) From the balance law of radial momentum (2.3*b*), we obtain

$$\rho_a r u^2 |_0^\infty + \left(\rho_a W \int_0^\infty r u \exp(-r^2/b^2)\,\mathrm{d}r\right)' + \int_0^\infty (r p_r - \rho_a v^2)\,\mathrm{d}r = 0. \tag{A 3}$$

The first term equals zero. Integrating the continuity, (2.3*a*), over r from $r = 0$ to an arbitrary value yields

$$ru = \frac{1}{2\rho_a}\left[\rho_a b^2 W\left(\exp\left(-\frac{r^2}{b^2}\right) - 1\right]\right..$$

Substituting this into the second term of (A 3) and integrating from $r = 0$ to $r \to \infty$ yields

$$\int_0^\infty (r\rho_a uw)_z\,\mathrm{d}r = \frac{(1-\surd 2)\,\pi^{1/2}}{3\surd 2}[W(\rho_a b^3 W)']'. \tag{A 4}$$

For the last terms of (A 3) we have

$$\int_0^\infty r p_r\,\mathrm{d}r = -\sigma \int_0^\infty r \frac{\mathrm{d}}{\mathrm{d}r}(\exp(-r^2/b^2))\,\mathrm{d}r = \frac{2\sigma}{b^2}\int_0^\infty r^2 \exp(-r^2/b^2)\,\mathrm{d}r = \frac{\pi^{1/2}}{2}\sigma b, \tag{A 5a}$$

$$\rho_a \int_0^\infty v^2\,\mathrm{d}r = \rho_a \int_0^\infty (v_1^2 + 2v_1 v_2 + v_2^2)\,\mathrm{d}r = \rho_a\left\{\Gamma^2 \int_0^\infty \frac{(1-\exp(-r^2/B^2))^2}{r^2}\,\mathrm{d}r\right.$$

$$+\frac{2\Gamma V}{b}\int_0^\infty \left(\exp\left(-\frac{r^2}{b^2}\right) - \exp\left(-\left(\frac{1}{B^2}+\frac{1}{b^2}\right)\right)\right)\mathrm{d}r + \frac{V^2}{b^2}\int_0^\infty r^2 \exp\left(-\frac{2r^2}{b^2}\right)\mathrm{d}r$$

$$= \pi^{1/2}\rho_a\left[\frac{\Gamma^2}{B}(2-\surd 2) + \Gamma V\left(1 - \frac{B}{(B^2+b^2)^{1/2}}\right) + \frac{bV^2}{8\surd 2}\right], \tag{A 5b}$$

with
$$V = -4\Gamma b/(B^2+b^2). \tag{A 6}$$

Substituting (A 4)–(A 6) into (A 3) yields the following formula for the pressure deficit:

$$\sigma b^2 = 2\rho_a\left[(2-\surd 2)\frac{b}{B}\Gamma^2 + \Gamma V b\left(1 - \frac{B}{(B^2+b^2)^{1/2}}\right) + \frac{b^2 V^2}{8\surd 2}\right] + \frac{b(\surd 2 - 1)}{2\surd 2}[W(\rho_a b^3 W)']'. \tag{A 7}$$

According to estimates by Carrier *et al.* (1985) the z-dependence of W is sufficiently weak, hence it is possible to ignore the last term in (A 7), and we obtain (2.5*d*), which differs from that given by Carrier *et al.* (1985). The reason for this difference is as follows: Carrier *et al.* integrated the condition $p_r = \rho_a v^2/r$ over r from $r = 0$ to $r \to \infty$, while it is necessary to calculate the integral

$$\int_0^\infty (r p_r - \rho_a v^2)\,\mathrm{d}r.$$

(v) When considering the equation of azimuthal angular momentum (2.3*c*), we make the following assumption. The response v_1 to the external circulation $\Gamma(z)$ changes faster in time than does the redistribution v_2 along the z-axis, that is $(v_2)_t \ll (v_1)_t''$, and the z-dependence is weak compared with the r-dependence. Hence, we assume $(v_2)_t \ll (v_1)_t, v_z \ll v_r$. Then we substitute (2.4) into (2.3*c*), integrate over r, from

$r = 0$ to $r \to \infty$, and decompose the expression obtained into two parts according to the orders of magnitude of the various terms and obtain

$$\left[\rho_a W \int_0^\infty \left(r\Gamma\left(1-\exp\left(\frac{-r^2}{B^2}\right)\right)+\left(\frac{V}{b}\right)r^3\exp\left(\frac{-r^2}{b^2}\right)\right)\mathrm{d}r\right]' = 0, \tag{A 8}$$

$$2\Gamma B^{-3}B_t \int_0^\infty r^3\exp\left(\frac{-r^2}{B^2}\right)\mathrm{d}r+\rho_a u\left(r\Gamma\left(1-\exp\left(\frac{-r^2}{B^2}\right)\right)\right.$$
$$\left.+\left(\frac{V}{b}\right)r^3\exp\left(\frac{-r^2}{b^2}\right)\right)\Bigg|_0^\infty = 0. \tag{A 9}$$

Formula (A 8) yields

$$\rho_a b^2 W\left(\frac{\Gamma b^2}{B^2+b^2}+\tfrac{1}{4}Vb\right) = \psi(t).$$

Since $\lim_{r\to\infty} W(z,t) = 0$, we have $\psi(t) = 0$ and thus obtain (2.5*e*). Formula (A 9) yields $BB_t = -\alpha b W$, or (2.5*f*).

This argument can even be made stronger. If in the equation for azimuthal angular momentum we do not ignore $(v_2)_t$ as before, but only the terms proportional to the time derivative of the plume size b_t, we obtain

$$v_t = -2r\Gamma B^{-3}B_t\exp(-r^2/B^2)+b^{-1}rV_t\exp(-r^2/b^2).$$

Making use of (2.5*e*) at $b_t = 0$ yields

$$v_t = -rB^{-4}\Gamma\exp(-r^2/B^2)-4(1+b^2/B^2)\exp(-r^2/b^2)(B^2)_t.$$

Substituting this expression into (2.3*c*) and integrating over r from $r=0$ to $r=\infty$ yields the equation

$$(1-4(B^2/b^2+1)^{-2})(B^2)_t = -2\alpha b W \tag{A 10}$$

instead of (2.5*f*). The functions

$$t_{new} = \frac{b}{2\alpha W}(C_{new}-(B^2/b^2)-4b^2(B^2+b^2)^{-1})$$
$$t_{old} = b/(2\alpha W)(C_{new}-B^2/b^2)$$

are the solutions of (A 10) and (2.5*f*), respectively, and we shall choose $C_{new} = C_{old} = C$ at $B(0)/b \gg 1$. It then follows that

$$\delta = \frac{t_{old}-t_{new}}{t_{old}} = 4\left(C-\frac{B^2}{b^2}\right)^{-1}\left(1+\frac{B^2}{b^2}\right)^{-1}$$

for the relative error as a function of B/b. At the ground level we thus obtain

$B/b = 2.5$	2	1.5	1.2	1.7	1,
$\delta/10^{-3} = 2.5$	4.3	6.6	8.8	9.7	10.

Errors are decreasing with growing B/b. Of course the model is only valid for $B > b$, and this justifies the assumption $(v_2)_t \ll (v_1)_t$.

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